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DEFLECTIONS OF A CIRCULAR BEAM OUT OF ITS INITIAL PLANE

by Enrico Volterra

ENGINEERING MECHANICS DIVISION

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PAPERS

DEFLECTIONS OF A CIRCULAR BEAM OUT OF ITS INITIAL PLANE

By Enrico Volterra[1]

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[2] Numbers in parentheses refer to the Bibliography at the end of the paper.

Synopsis

The Saint-Venant equations (1)[2] for a circular beam bent out of its plane of initial curvature are applied to the study of the deflections of beams subjected to uniformly distributed forces and supported at symmetric points, and of the deflections of circular-arc bow girders supporting distributed or concentrated forces. The solutions are given in explicit form. Curves of the deflections, angles of twist, and bending and twisting moments are presented for certain cases.

Introduction

Notation.—The letter symbols in this paper are defined where they first appear, in the text or by diagram, and are assembled alphabetically in Appendix I.

In previous papers (2), the Saint-Venant equations for circular beams bent out of their plane of initial curvature have been applied to the study of deflections of beams resting on elastic foundations and loaded by concentrated symmetric forces. The same equations can also be applied to the study of the deflections of circular beams subjected to uniformly distributed loads and supported at equidistant points (see Fig. 1) and to the study of the deflections of circular-arc bow girders with uniformly distributed or concentrated loads (see Fig. 2). The solution can be expressed in explicit form and the final results presented in tabular form.

The problem of the bending of beams of simple or double curvature is important in strength of materials. Many authors have contributed to this field of research. Among them are B. de Saint-Venant (1), H. Resal (3), B. G. Kannenberg (4), H. Marcus (5), (6), F. Düsterbehn (7), F. Schleicher (8), C. B. Biezeno (9), A. J. S. Pippard and F. L. Barrow (10), (11), and more recently M. B. Hogan (12), (13), (14), (15), (16), B. B. Moorman and M. B. Tate (17) and B. Velutini (18).

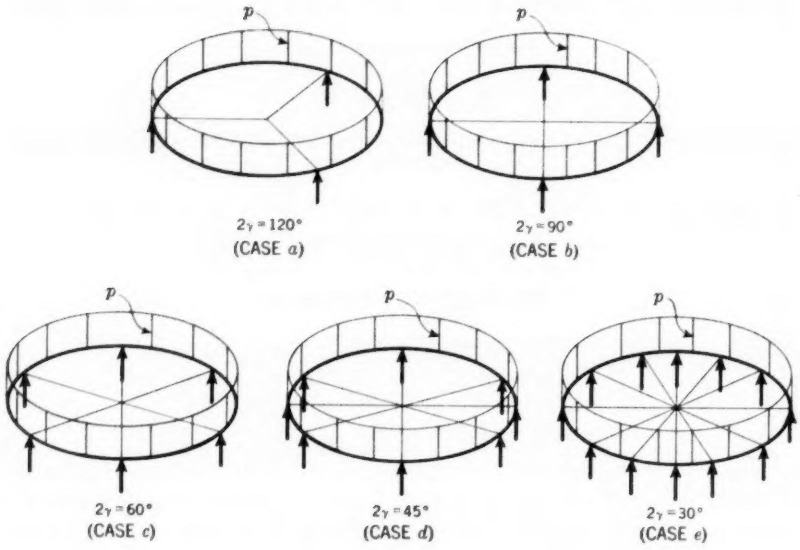


FIG. 1.—CIRCULAR BEAMS UNIFORMLY LOADED

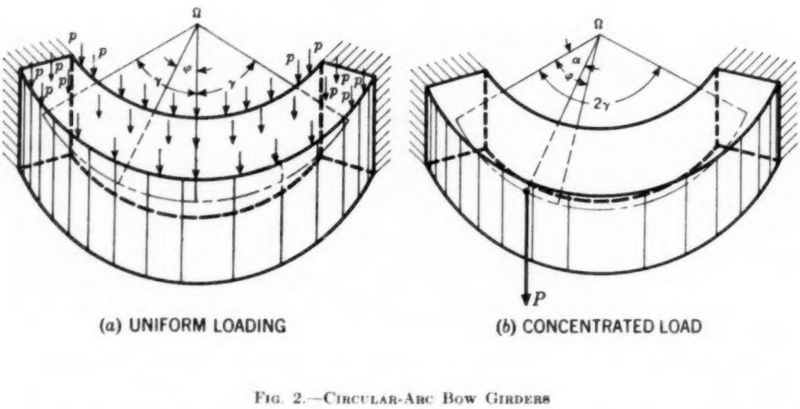


FIG. 2.—CIRCULAR-ARC BOW GIRDERS

Many of the methods proposed are derived from considerations of strain energy; others are graphical in nature. In the particular case of a circular beam bent out of its plane of initial curvature, it is possible in the most common cases encountered in practice—especially in that of reinforced concrete structures—to express the final results in a simple explicit form. Tables can provide the engineer with the required data, thus reducing the labor involved in computations.

The object of the paper is to demonstrate this fact, and for this reason every case considered is followed by a specific numerical example to show how the corresponding tables can be used.

Bending of a Curved Beam out of Its Plane of Initial Curvature

A curved beam, built-in at one end, having constant inertia characteristics is to be considered (see Fig. 3). The beam will be referred to a system of rectangular coordinates x, y, and z with the origin O at the centroid of the cross section, the axes x and y coinciding with the principal axes of inertia of the cross section and the axis z with the tangent to the center line at O. The plane xz coincides with the horizontal plane of the initial curvature of the beam, with the positive direction of x toward the center of curvature, y directed positive downward, z taken positive away from the built-in end, and the arc s of the center line measured from the fixed end.

The displacement of the centroid O during bending is resolved into three components, u, v, and w, in the directions of the x, y, and z axes, respectively. The angle of twist β of the cross section about the z-axis is considered positive when the rotation is counter-clockwise with respect to the z-axis. The deformation of an element of the curved bar, bounded by two adjacent cross sections, consists of bending in the principal planes xz and yz and twist about the z-axis. If M_x, M_y, and M_z are the moments on the cross section at O about the x-axis, y-axis, and z-axis, respectively (their positive directions are shown in Fig. 3), $E\,I_x$ and $E\,I_y$ the flexural rigidity about the axes indicated by subscripts, K the torsional rigidity, and $\frac{1}{R}$ the initial curvature of the center line of the bar, the equations for determining the changes in curvature $\left(\frac{1}{R_1}; \frac{1}{R_2}\right)$ and the twist are (19) (20)

and

$$\left.\begin{aligned} M_x &= \frac{E\,I_x}{R_1} \\ M_y &= E\,I_y\left(\frac{1}{R_2} - \frac{1}{R}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

In these expressions, E is the modulus of elasticity, I the moment of inertia, and R the radius of curvature. The expressions for the curvatures and twist as functions of u, v, w and β are, (19) (20)

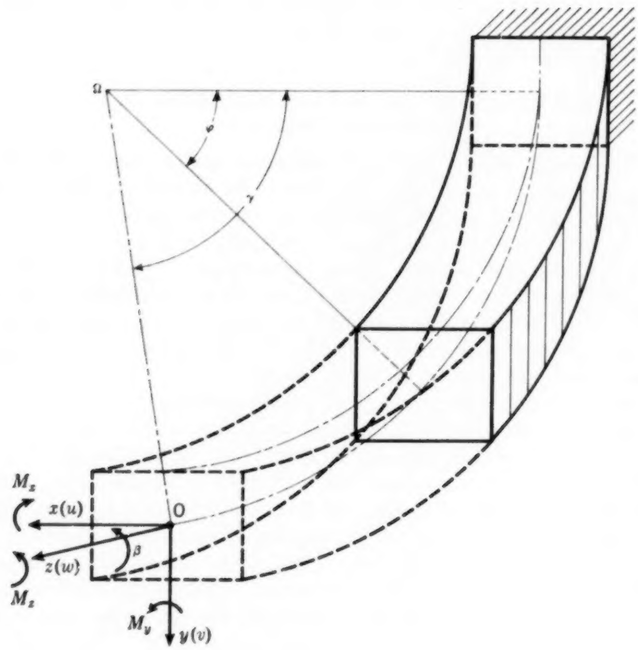


FIG. 3.—DIAGRAM SHOWING SYMBOLS AND ORIENTATION

$$\left.\begin{aligned}\frac{1}{R_1} &= \frac{\beta}{R} - \frac{d^2v}{ds^2}\\ \frac{1}{R_2} &= \frac{1}{R} - \frac{u}{R^2} + \frac{d^2u}{ds^2}\\ \theta &= \frac{d\beta}{ds} + \frac{1}{R}\frac{dv}{ds}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The condition that the length of the axis of the bar remain unchanged during bending and twist is expressed by (19) (20)

$$\frac{dw}{ds} - \frac{u}{R} = 0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

By the use of Eqs. 1, 2, and 3, the displacement u, v, w, and the angle β can be determined in the case of small deflections of curved beams.

The particular cases to be explained are (1) a circular beam subjected to an uniformly distributed load and supported at symmetric points, (2) a circular-arc bow girder with a distributed load, and (3) a circular-arc bow girder with a concentrated load. In these cases, the displacements u and w can be neglected and Eqs. 1 and 2 reduce to the two equations,

$$\left.\begin{aligned}M_x &= E\,I_x\left(\frac{\beta}{R} - \frac{d^2v}{ds^2}\right)\\ M_z &= K\,\theta = K\left(\frac{d\beta}{ds} + \frac{1}{R}\frac{dv}{ds}\right)\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Circular Beam Subjected to a Uniformly Distributed Load and Supported Symmetrically.—If p is the uniformly distributed load, per unit length of the beam, if $2\,\gamma$ is the angular distance between the points of support (Fig. 1), and with the following notation,

$$\left.\begin{aligned}Y &= \frac{v}{R}\\ a &= \frac{E\,I_x}{R}\\ \mu &= \frac{K}{E\,I_x}\\ \nu &= \frac{p\,R^3}{E\,I_x}\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

Eqs. 4 become

$$\left.\begin{aligned}a\left(\beta - \frac{d^2y}{d\phi^2}\right) &= M_x\\ \mu\,a\left(\frac{d\beta}{d\phi} + \frac{dy}{d\phi}\right) &= M_z\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

For equilibrium, however, the equations,

$$\left.\begin{aligned}\frac{dM_z}{d\phi} - M_x &= 0\\ \frac{d^2M_x}{d\phi^2} + \frac{dM_z}{d\phi} - a\,\nu &= 0\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

must be verified with the boundary conditions:

$$\left.\begin{aligned}\beta\,(\gamma) &= \beta\,(-\gamma)\\ \frac{d\beta\,(\gamma)}{d\phi} &= \frac{d\beta\,(-\gamma)}{d\phi}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(8a)$$

$$\left.\begin{aligned}Y\,(\gamma) &= Y\,(-\gamma)\\ \frac{dY\,(\gamma)}{d\phi} &= \frac{dY\,(-\gamma)}{d\phi}\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(8b)$$

and

$$\left.\begin{aligned}\frac{d^2Y\,(\gamma)}{d\phi^2} &= \frac{d^2Y\,(-\gamma)}{d\phi^2}\\ \frac{dM_x\,(-\gamma)}{d\phi} - \frac{dM_x\,(\gamma)}{d\phi} &= 2\,a\,\gamma\,\nu\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(8c)$$

The angle ϕ is measured from the center between the points of support. From Eqs. 6, 7, and 8 are obtained:

$$\frac{\beta}{\nu} = \frac{\mu + 1}{2\,\mu}\left[2 - \frac{\gamma}{\sin\gamma}\,\phi\sin\phi - \frac{\gamma}{\sin\gamma}\,(1 + \gamma\cot\gamma)\cos\phi\right]\ldots\ldots(9a)$$

$$\frac{Y}{\nu} = \frac{1}{2\,\mu}\left\{\phi^2 - 2 + \frac{(\mu + 1)\,\gamma}{\sin\gamma}\,\phi\sin\phi\right.$$
$$\left. + \frac{\gamma}{\sin\gamma}\,[\mu + 3 + (\mu + 1)\,\gamma\cot\gamma]\cos\phi + A'\right\}\ldots(9b)$$

$$\frac{M_x}{a\,\nu} = 1 - \frac{\gamma}{\sin\gamma}\cos\phi\ldots\ldots\ldots\ldots\ldots\ldots(9c)$$

and

$$\frac{M_z}{a\,\nu} = \phi - \frac{\gamma}{\sin\gamma}\sin\phi\ldots\ldots\ldots\ldots\ldots\ldots(9d)$$

In the case of rigid supports $Y = 0$ for $\phi = \gamma$, and the constant of Eq. 9*b* assumes the value,

$$A' = -\frac{1}{2\,\mu}\,\{\gamma^2 - 2 + (\mu + 1)\,\gamma^2 + \gamma\cot\gamma\,[\mu + 3 + (\mu + 1)\,\gamma\cot\gamma]\}\ldots(10)$$

The values of the functions β, Y, M_x and M_z are given in Figs. 4(*a*), 5, and 6(*a*) for the following values of the parameters, $\mu = 0.7$ and 1; and $2\,\gamma = 120°$, 90°, and 45°. In the computation of the numerical scales, the constant A' was taken as equal to zero.

TABLE 1.—COMPUTATIONS FOR THE MOMENTS IN A UNIFORMLY LOADED CIRCULAR BEAM[a]

ϕ, in degrees	β, in radians	Y, in inches	M_x, in foot-tons	M_z, in foot-tons
0	−0.000812	0.005616	3.4875	0
5	−0.000644	0.005064	2.9683	0.2888
10	−0.000191	0.003360	1.4110	0.4872
15	0.000391	0.001680	−1.1700	0.5045
20	0.000847	0.000820	−4.7533	0.2542
22.5	0.000924	0	−6.9137	0

[a] See Fig. 5.

Numerical Example.—Table 1 contains computations of moments in a circular beam having a radius of $R = 20$ ft. The beam is assumed to be of reinforced concrete with a modulus of elasticity of 3×10^6 lb per sq in. The beam is assumed to have a square cross section (16 in. by 16 in.), to rest on eight rigid supports, and to support a load, p, of 0.5 tons per ft.

The following values are computed: $I_x = 5{,}451$ in.^{4}, $a = 5{,}688.54$ ft-tons, $\nu = 0.0239$, $a\,\nu = 133.11$ ft-tons, and $\mu = 0.7$. Fig. 5 is applicable in solving this problem.

Circular-Arc Bow Girder With Distributed Load.—Eqs. 5, 6, and 7 are valid also in this case (see Fig. 2) provided that the following boundary conditions are assumed:

$$\beta\ (\gamma) = \beta\ (-\ \gamma) = Y\ (\gamma) = Y\ (-\ \gamma) = \frac{dY\ (\gamma)}{d\phi} = \frac{dY\ (-\ \gamma)}{d\phi} = 0 \quad (11)$$

The solutions of Eqs. 5, 6, 7, and 11 are

$$\frac{\beta}{\nu} = \frac{\mu + 1}{2\,\mu}\left[2 + D_1\,\phi \sin\phi - \cos\phi\left(\frac{2}{\cos\gamma} + D_1\,\gamma \tan\gamma\right)\right] \quad (12a)$$

$$\frac{Y}{\nu} = \frac{1}{2\,\mu}\left[\phi^2 - \gamma^2 - 2\ (\mu + 1)\left(1 - \frac{\cos\phi}{\cos\gamma}\right)\right]$$
$$+ \frac{D_1}{2\,\mu}\left\{-\ (\mu + 1)\ \phi \sin\phi + [(\mu + 1)\ \gamma \tan\gamma - 2] \cos\phi + 2 \cos\gamma\right\} \quad (12b)$$

$$\frac{M_x}{a\,\nu} = 1 + D_1 \cos\phi \quad (12c)$$

$$\frac{M_z}{a\,\nu} = \phi + D_1 \sin\phi \quad (12d)$$

in which:

$$D_1 = 2\,\frac{\gamma \cos\gamma - (1 + \mu) \sin\gamma}{(1 + \mu)\ \gamma + (\mu - 1) \sin\gamma \cos\gamma} \quad (13)$$

In the particular case in which $\gamma = 90°$,

$$D_1\,\gamma \tan\gamma + \frac{2}{\cos\gamma} = \frac{\pi^2 + 4\ (\mu - 1)}{\pi\ (1 + \mu)} \quad (14)$$

The values of the functions β, Y, M_x, and M_z are given in Figs. 4(*b*), 6(*b*), and

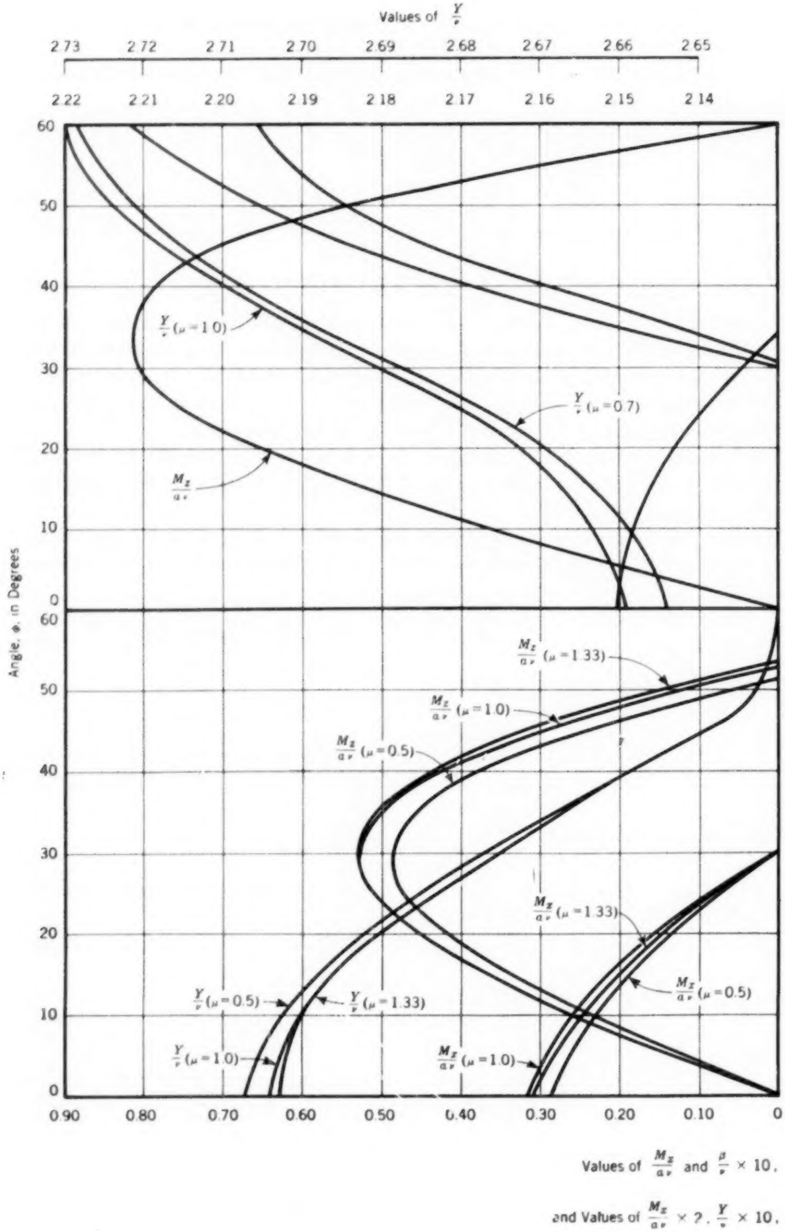
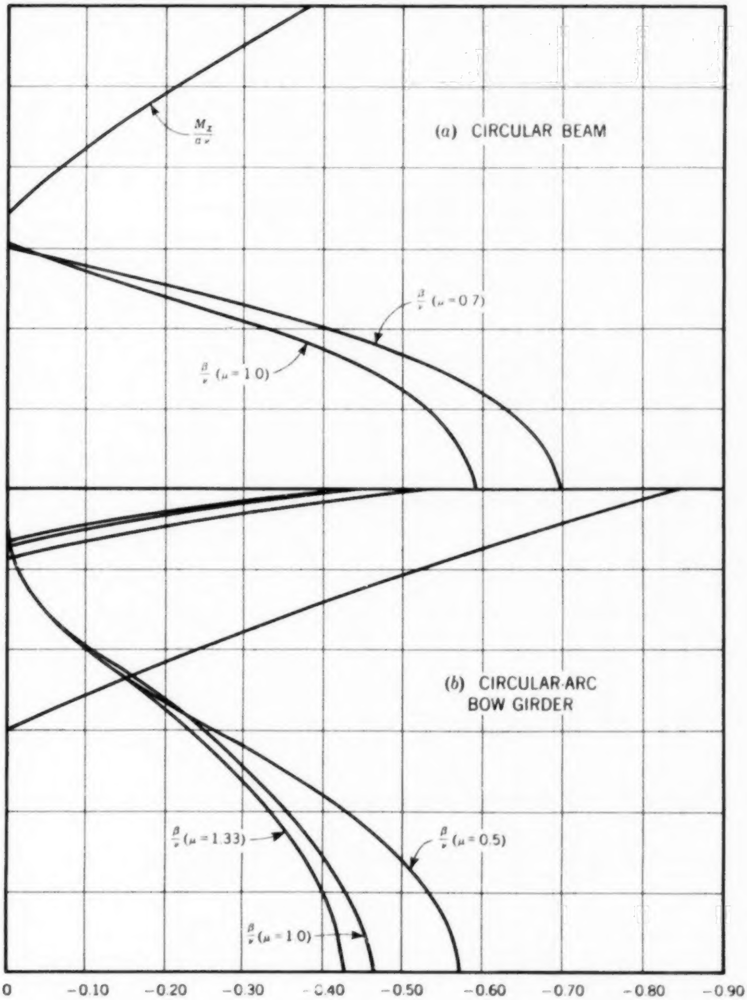


FIG. 4.—FUNCTIONS FOR

[for $\mu = 0.7$ in Fig. 4(a)]

[for $\mu = 1.0$ in Fig. 4(a)]



Each in Fig. 4(a); Values of $\frac{M_z}{a\nu} \times 10$;

and $\frac{\beta}{\nu} \times 5$, Each in Fig. 4(b)

UNIFORM LOAD, $2\gamma = 120°$

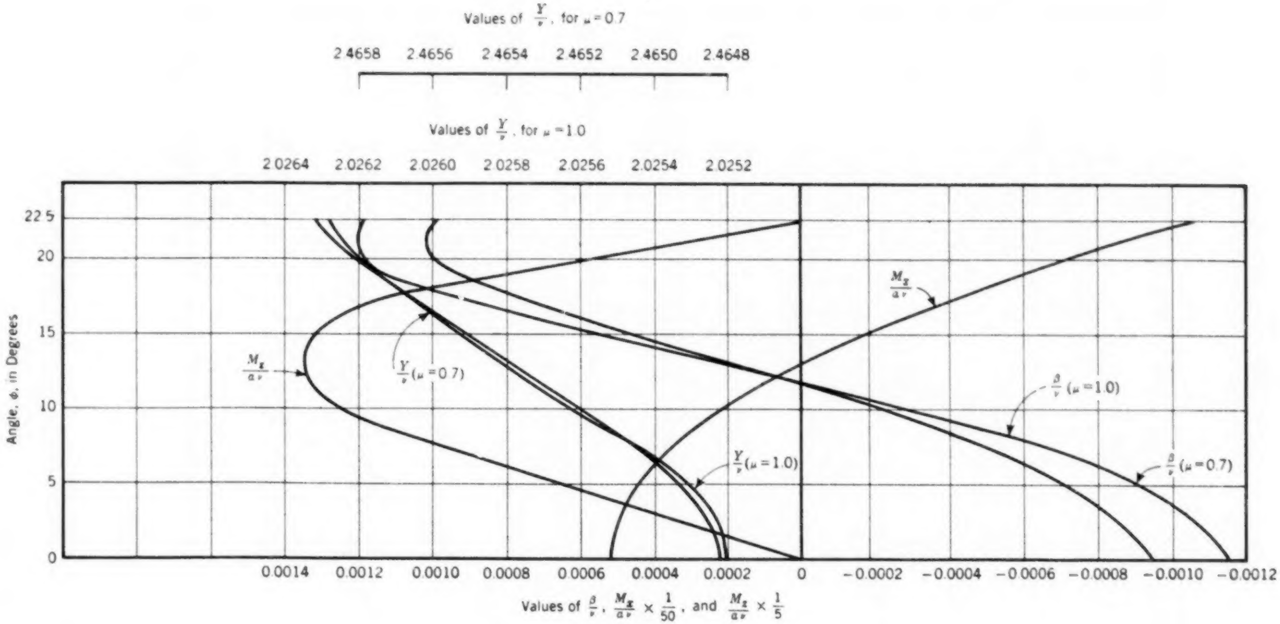


FIG. 5.—FUNCTIONS FOR A CIRCULAR BEAM WITH A DISTRIBUTION LOAD
($2\gamma = 45°$, AND $\mu = 0.7$ AND 1)

7 for the following values of the parameters: $\mu = 0.5$, 1, and 1.33; and $2\gamma = 120°$, 90°, and 45°.

Numerical Example; Bow Girder.—A circular bow girder is assumed to be subjected to a uniformly distributed load, p, of 0.25 tons per ft. The beam is of reinforced concrete ($E = 3 \times 10^6$ lb per sq in.), and its rectangular cross section is 20 in. wide and 12 in. deep. The angular distance between the built-in ends is $2\gamma = 120°$.

The following values are computed: $I_x = 8 \times 10^3$ in.^{4}, $a = 8{,}333.33$ ft-tons, $\nu = 3 \times 10^{-3}$, $a\,\nu = 25$ ft-tons and $\mu = 0.5$. The computations for this example are in Table 2, and are based on Fig. 4(*b*).

TABLE 2.—COMPUTATIONS FOR THE MOMENTS IN A UNIFORMLY LOADED CIRCULAR-ARC BOW GIRDER[a]

ϕ, in degrees	β, in radians	v, in inches	M_x, in foot-tons	M_z, in foot-tons
0	0.0003468	0.02415	3.6290	0
10	0.0003222	0.02275	3.1940	0.6080
20	0.0002535	0.01881	1.9025	1.0650
30	0.0001572	0.01315	− 0.2065	1.2245
40	0.0000588	0.007030	− 3.0690	0.9490
50	0.00000702	0.002052	− 6.5977	0.1145
60	0	0	−10.6855	−1.3865

[a] See Fig. 4(*b*)

Circular-Arc Bow Girder With Concentrated Load.—If P is the concentrated load, 2γ the angular distance between the extremities of the beam, and α the angular distance between the end of the beam ($\phi = 0$) and the point where the concentrated force is applied (see Fig. 3), it follows that $\nu = \dfrac{P\,R^2}{E\,I_x}$ and that Eqs. 6 are still valid. For equilibrium, the following equations must be verified:

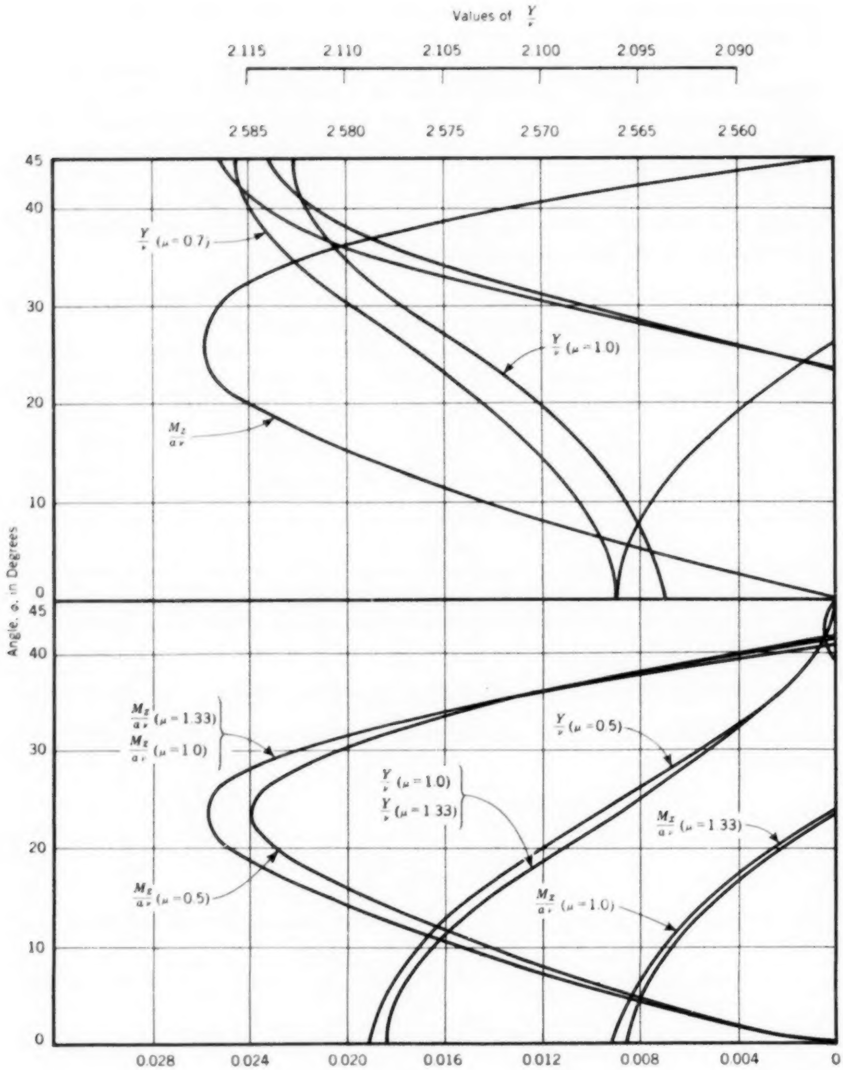
$$\left.\begin{aligned} \frac{dM_z}{d\phi} - M_x &= 0 \\ \frac{d^2M_x}{d\phi^2} + \frac{dM_z}{d\phi} &= 0 \end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

The boundary conditions and the conditions of continuity are, in this case,

$$\beta\,(0) = \beta\,(2\gamma) = Y\,(0) = Y\,(2\gamma) = \frac{dY\,(0)}{d\phi} = \frac{dY\,(2\gamma)}{d\phi} = 0\ldots(16a)$$

$$\frac{dM_x\,(\alpha + 0)}{d\phi} - \frac{dM_x\,(\alpha - 0)}{d\phi} = -\,a\,\nu\ldots\ldots\ldots\ldots(16b)$$

and β, $\dfrac{d\beta}{d\phi}$, Y, $\dfrac{dY}{d\phi}$, and $\dfrac{d^2Y}{d\phi^2}$ are all continuous for $\phi = a$. Because of the discontinuity at $\phi = \alpha$, the analytical expression for the bending and twisting moments M_x and M_z for the displacement v and the angle of twist β are different in the intervals $0, \alpha$ and $\alpha, 2\gamma$. If $\beta = \bar{\beta}$; $y = \bar{y}$; $M_x = \bar{M}_x$; and $M_z = \bar{M}_z$; for $0 \leq \phi \leq \alpha$ and $\beta = \bar{\beta} + \beta^*$, $Y = \bar{Y} + Y^*$, $M_x = \bar{M}_x + M^*_x$; $M_z = \bar{M}_z + M^*_z$ for $\alpha \leq \phi \leq 2\gamma$, the solution of Eqs. 6, 15, and 16 is



Values of $\frac{M_z}{a\nu} \times \frac{4}{5}$ and $\frac{M_x}{a\nu} \times \frac{2}{25}$,

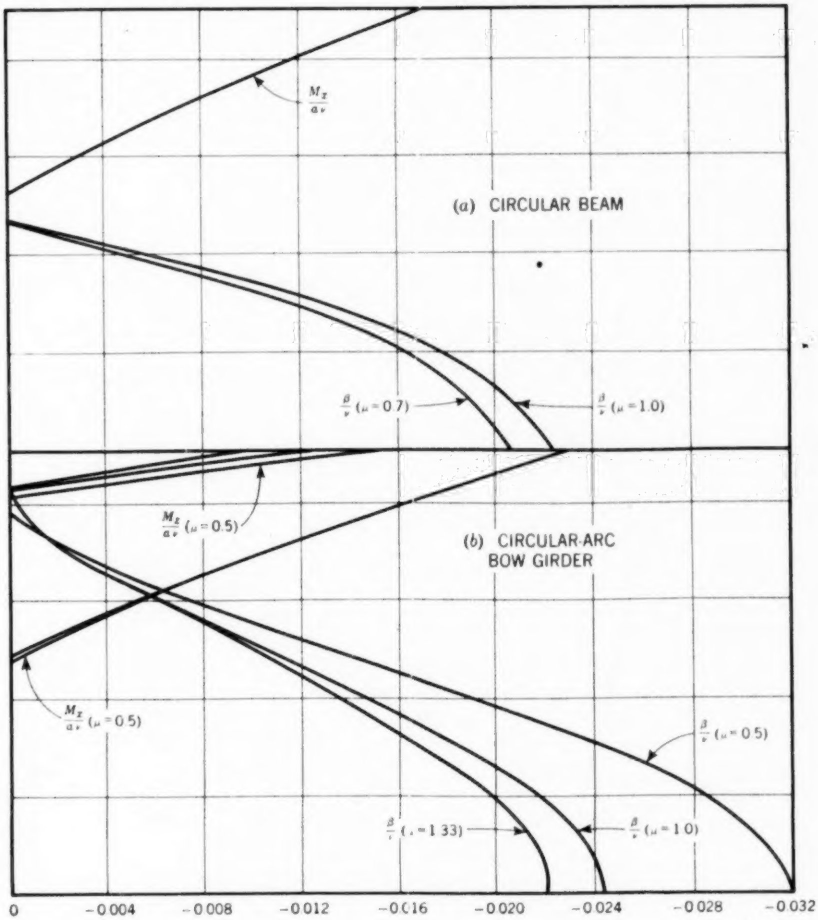
Values of $\frac{M_z}{a\nu}$, $\frac{M_x}{a\nu} \times \frac{1}{10}$, $\frac{Y}{\nu}$,

Values of $\frac{\beta}{\nu}$ for μ = 0.7 in Fig. 6(a);

FIG. 6.—FUNCTIONS FOR

[for $\mu = 1.0$ in Fig. 6(a)]

[for $\mu = 0.7$ in Fig. 6(a)]



Each in Fig. 6(a);

and $\frac{\beta}{\nu} \times \frac{4}{5}$, Each in Fig. 6(b);

and Values of $\frac{\beta}{\nu} \times \frac{4}{3}$ for $\mu = 1.0$ in Fig. 6(a)

UNIFORM LOAD, 2 = 90°

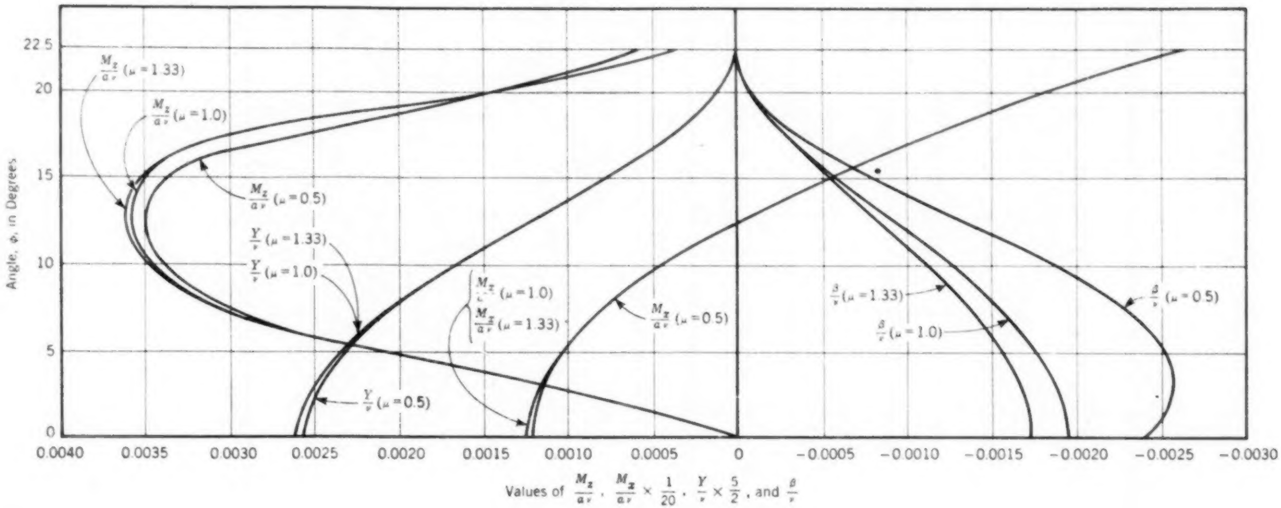


FIG. 7.—FUNCTIONS FOR A CIRCULAR-ARC BOW GIRDER WITH A DISTRIBUTED LOAD
($2\gamma = 45°$; AND μ = 0.5, 1.0, AND 1.33)

$$\beta^* = \frac{\mu + 1}{2\mu} \nu [(\phi - \alpha) \cos (\phi - \alpha) - \sin (\phi - \alpha)] \quad \ldots\ldots (17a)$$

$$Y^* = - \frac{\nu}{2\mu} [2 (\phi - \alpha) + (\mu + 1) (\phi - \alpha) \cos (\phi - \alpha) - (\mu + 3) \sin (\phi - \alpha)] \quad \ldots (17b)$$

$$M^*_x = - a \nu \sin (\phi - \alpha) \quad \ldots\ldots (17c)$$

and

$$M^*_z = - a \nu [1 - \cos (\phi - \alpha)] \quad \ldots\ldots (17d)$$

$$\bar{\beta} = \nu A (\mu + 1) \phi \sin \phi + \nu B [- (\mu + 1) \phi \cos \phi + (\mu - 1) \sin \phi] + 2 \nu C \sin \phi \quad \ldots (18a)$$

$$\bar{Y} = \nu A [2 - (\mu + 1) \phi \sin \phi - 2 \cos \phi] + \nu B (\mu + 1) (\phi \cos \phi - \sin \phi) + 2 \nu C (\phi - \sin \phi) \quad \ldots (18b)$$

$$\bar{M}_x = 2 \mu \nu a (A \cos \phi + \beta \sin \phi) \quad \ldots\ldots (18c)$$

$$\bar{M}_z = 2 \mu \nu a (A \sin \phi - B \cos \phi + C) \quad \ldots\ldots (18d)$$

The constants A, B, and C are determined from the following equations:

$$A = [(\mu + 1) \gamma^2 - 2 + 2 \cos \gamma] \frac{L_1}{D_2} \sin \gamma + [- (\mu + 1) \gamma^2 \cos \gamma + (\mu - 1) \gamma \sin \gamma + 2 \sin^2 \gamma] \frac{L_2}{D_2} - (1 - \cos \gamma) \frac{L_3}{D_2} \quad \ldots (19a)$$

$$B = - [(\mu + 1) \gamma^2 \cos \gamma + (\mu - 1) \gamma \sin \gamma + 2 (1 - \cos \gamma)^2] \frac{L_1}{D_2} + [- (\mu + 1) \gamma^2 + 2 - 2 \cos \gamma] \frac{L_2}{D_2} \sin \gamma + \sin \gamma \frac{L_3}{D_2} \quad \ldots (19b)$$

$$C = - \sin \gamma \frac{L_1}{D_2} - (1 - \cos \gamma) \frac{L_2}{D_2} + [(\mu + 1) \gamma - (\mu - 1) \sin \gamma] \frac{L_3}{D_2} \quad \ldots (19c)$$

in which

$$D_2 = [(\mu + 1) \gamma + (\mu - 1) \sin \gamma] \Delta \quad \ldots\ldots (20)$$

$$\Delta = 4 (1 - \cos \gamma) - \gamma [(\mu + 1) \gamma - (\mu - 1) \sin \gamma] \quad \ldots\ldots (21)$$

$$L_1 = \frac{\mu + 1}{2\mu} [(\gamma - \alpha) \cos (\gamma - \alpha) - \sin (\gamma - \alpha)] \quad \ldots\ldots (22a)$$

$$L_2 = \frac{1}{2\mu} [(\mu + 1) (\gamma - \alpha) \sin (\gamma - \alpha) + 2 \cos (\gamma - \alpha) - 2] \quad \ldots (22b)$$

and

$$L_3 = \frac{1}{\mu} [\sin (\gamma - \alpha) - (\gamma - \alpha)] \quad \ldots\ldots (22c)$$

The values of the constants A, B, and C are given in Figs. 8, 9, 10, and 11 for the following values of the parameters: $\mu = 0.564$; 0.7; 0.85; 1.15; 1.33; $2\gamma = 120°$, $90°$, and $45°$, and for the angle ϕ varying in the interval $(0, \gamma)$ by $10°$.

Numerical Example: Bow Girder With Point Load.—A reinforced concrete ($E = 3 \times 10^6$ lb per sq in.) circular bow girder, 24 in. by 24 in. in cross section, is assumed to support a concentrated load, $P = 10$ tons, at the center. The angle between the built-in ends is $2\gamma = 90°$, so that P is at $\alpha = 45°$.

From this information, the following are computed: $I_x = 27{,}648$ in.^{4}, $\mu = 0.7$, $\nu = 0.013889$, $a = 14{,}400$ ft-tons, and $2\,\mu\,\nu\,a = 280$ ft-tons. From Fig. 10 are obtained: $A = -0.16$, $B = 0.345$, and $C = 0.357$. In Table 3 are the results obtained for this particular case by applying Eqs. 18.

TABLE 3.—COMPUTATIONS FOR THE MOMENTS IN A CIRCULAR-ARC BOW GIRDER WITH CONCENTRATED LOAD[a]

ϕ, in degrees	β, in radians	r, in inches	M_x, in foot-tons	M_t, in foot-tons
0	0	0	−44.7104	3.2872
5	−0.00002012	0.002544	−36.1116	−0.2408
10	−0.00004321	0.009888	−27.2356	−3.0072
15	−0.0001223	0.020520	−18.1552	−4.9840
20	−0.0002243	0.033168	− 8.9376	−6.1712
25	−0.0003358	0.046584	0.3500	−6.5464
30	−0.0004443	0.059328	9.6348	−6.1124
35	−0.0005371	0.070032	18.8468	−4.8664
40	−0.0006015	0.077376	27.9160	−2.8252
45	−0.0006257	0.080352	36.7696	0.0000

[a] See Fig. 10

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APPENDIX I. NOTATION

The following symbols, adopted for use in the paper, and for the guidance of discussers, conform essentially with American Standard Letter Symbols for Structural Analysis (ASA—Z10.8—1942), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942:

A, A' = constants;

$a = \dfrac{E\,I_x}{R}$;

B = a constant;

C = a constant;

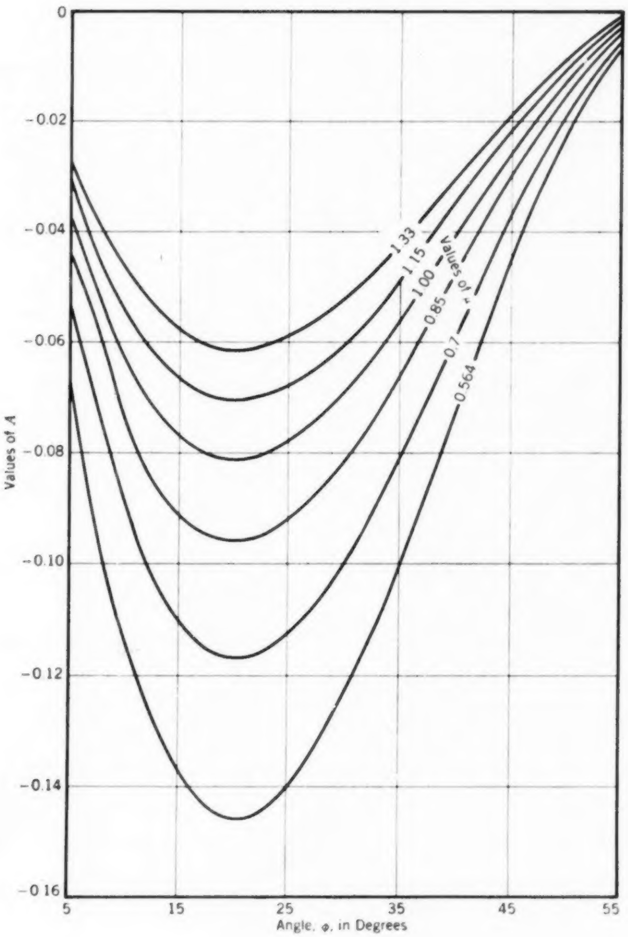

FIG. 8.—RELATIONSHIPS FOR A CIRCULAR-ARC BOW GIRDER WITH A CONCENTRATED LOAD, $2\gamma = 120°$

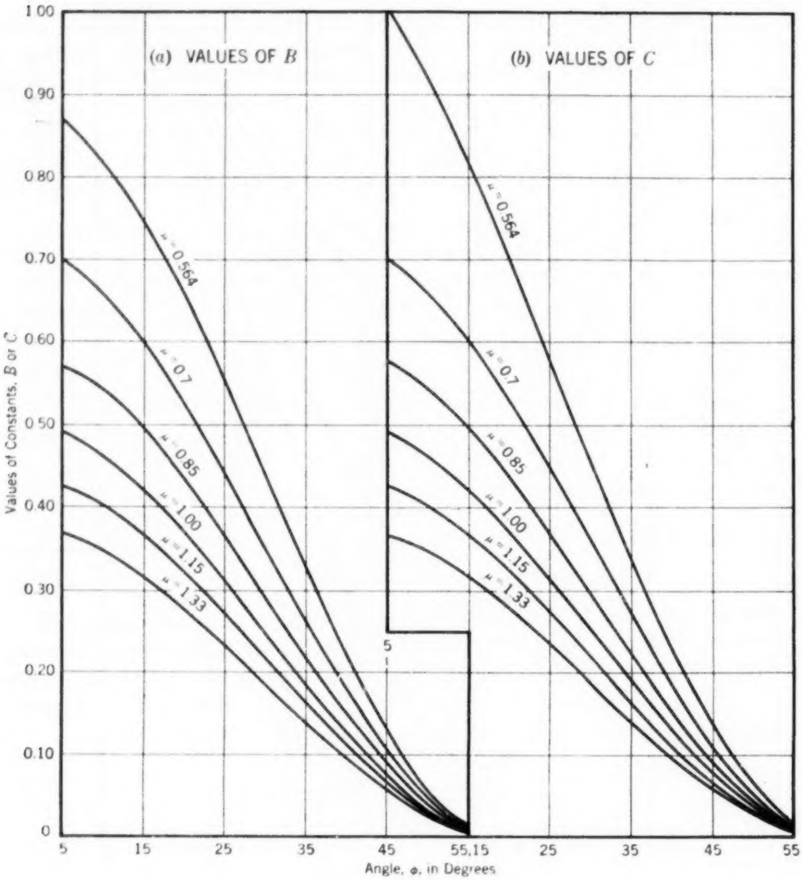

FIG. 9.—RELATIONSHIPS FOR A CONCENTRATED LOAD, $2\gamma = 120°$

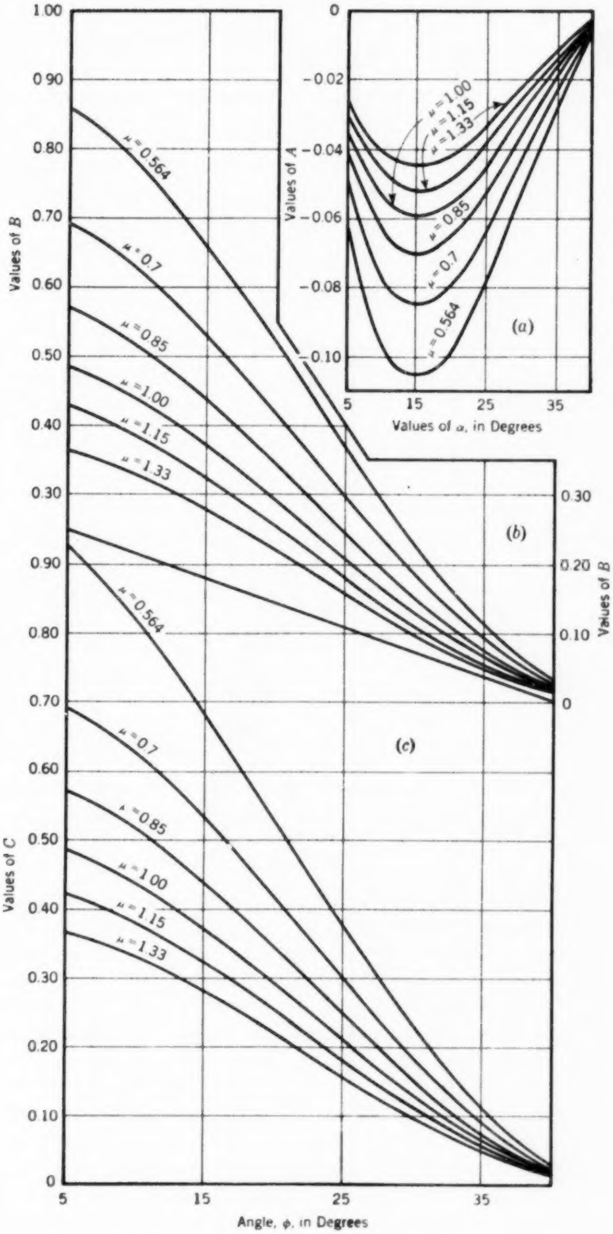

FIG. 10.—RELATIONSHIPS FOR A CONCENTRATED LOAD, $2\gamma = 90°$

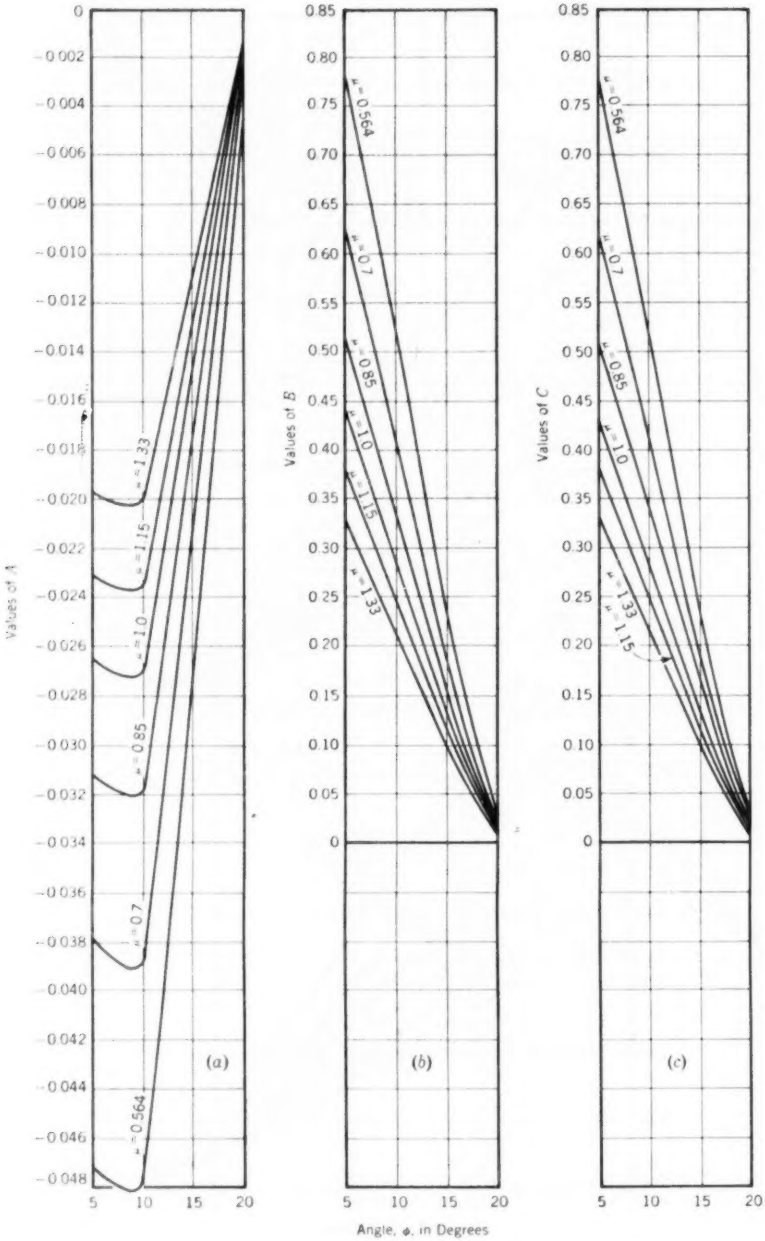

FIG. 11.—RELATIONSHIPS FOR A CONCENTRATED LOAD, $2\gamma = 45°$

D_1, D_2 = terms defined by Eqs. 13 and 20, respectively;
E = the modulus of elasticity;
I = the rectangular moment of inertia;
K = the torsional rigidity;
L_1, L_2, L_3 = terms defined by Eqs. 22;
M_y, M_x = bending moments;
M_z = twisting moments;
P = intensity of the concentrated load;
p = intensity of the distributed load;
R = the initial curvature of the center line of the beam;
s = arc length;
u, v, w = displacements along axes x, y, and z, respectively;
x, y, z = rectangular coordinate axes;
$Y = \frac{v}{R}$;
α = an angle defined by Fig. 2(b);
β = the angle of twist;
γ = one half the angular distance between points of support;
Δ = a term defined by Eq. 21;
θ = the angle of twist per unit length;
$\mu = \frac{K}{E\, I_x}$;
$\nu = \frac{P\, R^2}{E\, I_x}$;
ϕ = an angle defined by Fig. 2.

APPENDIX II. BIBLIOGRAPHY ON CURVED BEAMS

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